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Stochastic Resonance in Periodic Potentials

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We have studied the motion of a particle in a periodic potential plus a bias, driven by a noise and a coherent forcing. The response (power spectrum) of the particle at the driving forcing frequency is considered for different values of the noise intensity and of the bias. It is shown via direct simulation that the response displays the phenomenon of stochastic resonance, although the phenomenology is somehow different from the one observed in the standard bistable system.

KEY WORDS: Periodic potentials; stochastic resonance; analogue simulations; linear response theory.

1. INTRODUCTION

There is no doubt that periodic potentials are very important and very common in physical science. Many systems can be modeled by such a potential, and virtually any issue of a physics journal will contain some applications to real systems. It is impossible here to cite all the relevant literature, even having in mind the restricted field of nonlinear stochastic physics. Some of the fundamental ideas can be found in the book by Risken[(1)] or in recent reviews (Risken *et al.* ref. 2, Munakata *et al.* in ref. 3; see also ref. 4 for more recent references).

How would a system modeled by a periodic potential behave if we added a periodic excitation to the stochastic forcing? Although the problem in itself has been studied in a number of publications (see above references and also references in ref. 5), focusing attention on the mobility of the Brownian particle as function of the various parameters, it has never been approached before as a possible model where stochastic resonance could take place. The phenomenon of stochastic resonance, first employed to

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described geological glacial periods,[6–8] has recently attracted quite a lot of attention (e.g., refs. 9–26). Models where stochastic resonance has been observed are generally two-states model (but see the papers by M. Dykman *et al.* in this issue). For a very general review on stochastic resonance see also ref. 27. It is then very interesting to study whether the phenomenon is present even when there are more than two stable states.

2. THE MODEL

The model we have studied is described by the Langevin equation

$$\dot{x} = -\frac{\partial U(x)}{\partial x} + \xi(t) + A\cos(\omega t) \tag{1}$$

where $\xi(t)$ is a Gaussian noise with zero average and correlation

$$\langle \xi(t)\, \xi(s) \rangle = 2D\delta(t-s) \tag{2}$$

The potential chosen

$$U(x) = -\cos x - \alpha x \tag{3}$$

broadly speaking corresponds to the potential of a biased Josephson junction. We have finally the Langevin equation

$$\dot{x} = -\sin x + \alpha + \xi(t) + A\cos(\omega t) \tag{4}$$

There is a time-dependent Fokker–Planck equation corresponding to this Langevin equation, which reads

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left\{\sin x - \alpha - A\cos(\omega t) + D\frac{\partial}{\partial x}\right\} P(x,t) \tag{5}$$

In the "classical" approach to SR, one now considers the response of the signal $x(t)$ to a periodic modulation, looking at the Fourier transform at the driving frequency ω. However, in the case of periodic potential there could be some ambiguity in the definition of the response of the system to the periodic forcing. This is because the system can in principle diffuse very far from the origin [i.e., from the minimum located at $x = \sin^{-1}(\alpha)$].

We have considered here, then, the response of either $\sin(x)$ or $v(t) \equiv \dot{x}(t)$.

3. THE ANALOGUE SIMULATION

To simulate the model of Eq. (4) a simple electronic circuit has been built. Analogue simulations of a pendulum motion [which is related to the potential of Eq. (4)] are not new: the idea is to use a phase-locked loop.[28–32] We followed the same approach, improving the basic idea using minimum component techniques (see Fronzoni in ref. 33 and also ref. 34). Straight simulation of the potential via specialized electronic components can be carried out (see ref. 4 and also McClintock and Moss in ref. 33), but we found that for the case at hand the phase-locked loop technique was more reliable. The scheme of the electronic circuit is shown in Fig. 1. The sum of the currents at point A gives Eq. (4), which is the equation we want to study. For the noise input we used a home-made dichotomous noise generator (see Fronzoni in ref. 33 and also ref. 34). The low-pass filter between noise generator and circuit is chosen with a characteristic time constant much shorter than the integration time of the circuit, to make sure that the noise is perceived as white by the circuit. This is a necessary condition for a simple theoretical treatment of the problem.

Particular care must be devoted to the stability of the electronic circuit. Although in principle at the beginning of each acquisition run the parameters of the electronic circuit could be set with good precision, drifts of the oscillator and unavoidable offsets which develop with time tend to alter the various "constants" of the circuit. As a very rough check, we always monitored the phase space $\{\sin x, \dot{x}\}$ of the system on an oscilloscope. For particular values of the parameters (for instance, $\alpha = 0$) it is fairly easy to appreciate substantial deviations of the parameters from the preset values. Clearly, however, this is very much an empirical procedure: in practice we ran the simulations for some time and checked at the end of the run that the parameters had not drifted too much from their

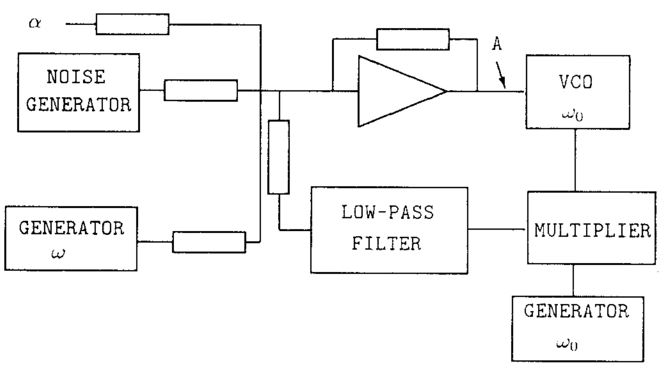

Fig. 1. Block diagram of the analogue circuit. The empty rectangular boxes are resistors.

initial value, discarding the acquired data if otherwise. The acquisition becomes critical as α approaches 1. This is the value ($\alpha = 1$) for which the potential becomes monotonic, and even small drifts of α can drastically change the behavior of the system.

We remind the reader that our final goal was to study the possible onset of SR in the model of Eq. (4). Instead of looking at the more traditional signal-to-noise ratio (SNR) in this model as function of the noise intensity, we decided to look at the signal at the driving frequency as the noise intensity is changed. There is no particular reason to do so, apart from the fact that if one was not interested at the background noise (necessary when the SNR is considered), it would be possible to obtain the amplitude and the phase relation of the signal with respect to the external driving in a most straightforward way. In our analogue simulation we took the various measurements sending the signal to a lock-in amplifier, driven by the external periodic forcing used in Eq. (4).

The first interesting result we found is that when one studies the response of $\sin(x)$ to the periodic forcing there seems to be no increase of the signal at the driving frequency as the noise intensity grows. This may look very strange: one would expect that at small enough noise intensities and/or for small biases a periodic potential should behave not differently from a bistable system, i.e., from a system where SR is well established. On the other hand, SR is a large-noise-intensity phenomenon: the net result is that for the noise intensities at which SR should set in the Brownian particle will be able to diffuse over many wells on the time scale of the periodic forcing, hence effectively removing any similarity with a bistable system. Also, we could say that $\sin(x)$ is a periodic function, hence it will not matter much whether a particle travels over many wells: as far as the SR seen in $\sin(x)$ is concerned, what really matters is that $\sin(x)$ has large excursions in phase with the external forcing. Obviously, even if this would be the case for small noise intensities, when the noise is large, there is no reason to expect that $\sin(x)$ and the external forcing should be strongly correlated.

On the other hand, if we were able to look at $x(t)$, we would expect to find a different result. We have already pointed out, however, that $x(t)$ is not very significant (and almost unmeasurable!); but $v(t) = \dot{x}(t)$ should have a behavior similar to the one shown by $x(t)$ when the noise is changed, given that in the Fourier transform $v(\omega) = i\omega x$. Focusing then on $v(t)$, as is clear form Fig. 2, the system does show an SR behavior as the noise intensity is changed. A more detailed discussion of the results will be presented in the next section.

4. RESULTS

In this section we present the experimental results. First, we will focus on the response to the periodic forcing observed in $v(t)$. The output of the lock-in amplifier yields the quantities

$$A_0 = \langle v(t) A \cos(\omega t) \rangle \tag{6}$$

$$B_0 = \langle v(t) A \sin(\omega t) \rangle \tag{7}$$

$$\tag{8}$$

from which we easily derive the amplitude of the response

$$S = (A_0^2 + B_0^2)^{1/2} \tag{9}$$

and the phase between the signal and the external forcing

$$-\phi = \tan^{-1} \frac{B_0}{A_0}$$

Figure 2 shows the response S at the driving frequency as the noise intensity is changed. For no bias we observed a monotonic increase to a limiting value as the noise is increased. It is clear from the figure that this limiting value does not depend on the bias, given that the data gathered for different biases fall on the same asymptotic curve. Intuitively, we can see why this behavior is to be expected. When the noise is very large the

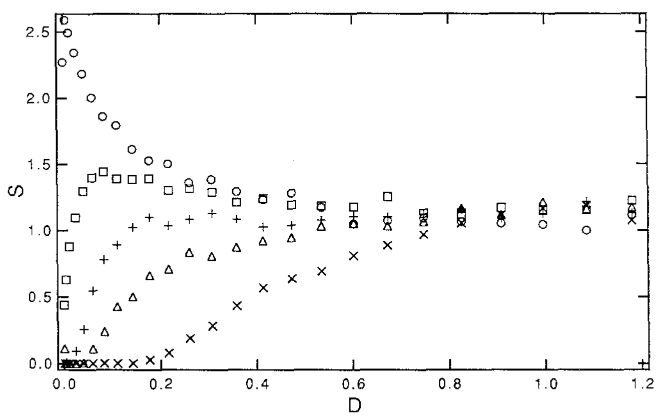

Fig. 2. Response (S) at the external driving frequency vs. noise intensity, for different bias values. $\alpha = (\times)$ 0.0, ($\triangle$) 0.5, ($+$) 0.66, ($\square$) 0.8, and ($\bigcirc$) 1.0.

particle feels the periodic potential only very little, and the motion is essentially described by

$$\dot{x} = \alpha + \xi(t) + A\cos(\omega t) \tag{10}$$

Averaging over the noise realizations, but in phase with the external periodic forcing, we have

$$\langle \dot{x} A\cos(\omega t)\rangle = \frac{A^2}{2} \tag{11}$$

$$\langle \dot{x} A\sin(\omega t)\rangle = 0 \tag{12}$$

which clearly shows that the response at the driving frequency ω, given by A, should be independent of both α and D.

The behavior at smaller noise intensities when the bias is different from zero is more complex. In particular, for larger biases the signal goes though a maximum before falling to its limiting value.

Another interesting quantity which can be measured in the system is the phase between the response and the periodic forcing. From Eq. (10) we have that the response should be in phase with the external forcing at large noise intensity: furthermore, this result should be independent of the bias. If we go back to the response of $x(t)$, remembering the phase relation between x and $\dot{x}$, we expect $x(t)$ to be quadrature to the external forcing. Figure 3 clearly shows that our conjecture is indeed correct: we observe that the phase between $\dot{x}$ and the external forcing goes from $\pi/2$ for small noise intensities to zero for large noise intensity. This transition takes place for noise intensities smaller than those at which the signal reaches its limiting value.

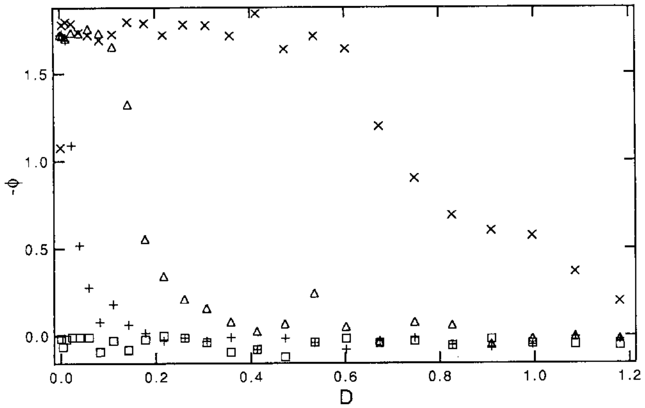

Fig. 3. Phase lag between signal and external periodic forcing vs. noise intensity for different α values Symbols as in Fig. 3.

Some comments and comparisons with the results found in the standard bistable Duffing oscillator are due. In the Duffing oscillator it has been shown[25] that the phase between the external forcing and the signal starts from small (near zero) values at small noise intensities (for small driving frequencies), decreases to a minimum which is a sizable fraction of $-\pi/2$ at intermediate noise intensities, and then increases again and reaches zero for large noise intensities (for other work and comments on the phase lag see also refs. 8, 13, 18–19, and 35). The intermediate values of the noise intensity correspond, roughly, to the value for which the SR phenomenon starts to set in. In our system, on the other hand, we have found that x starts from value close to zero for small noise intensities, then decreases *monotonically* to the limiting value of $-\pi/2$ as the noise is increased. Also, the limiting value is reached for noise intensities noticeably smaller than the values for which the signal starts to increase. The particular behavior of the phase observed here seems to suggest that the onset of SR in the system is somehow due to a mechanism different from the standard one, but perhaps analogous to the one observed in systems where the equilibrium populations of the stable state are different.[35–37]

5. THEORETICAL CONSIDERATIONS

A very useful tool for understanding and predicing the onset of SR is the interpretation in terms of linear response theory (LRT): within this approach one interprets the phenomenon starting from the equilibrium (i.e., without the periodic forcing) correlation functions (see the papers by Dykman *et al.* in the present issue and also refs. 23–25). We will show that, on the other hand, in this case the agreement between theory and simulations is far from satisfactory. More theoretical work is required, given that it is not even established that LRT can be applied to the case at hand.

Within the LRT approach to SR we can easily write the distribution in the presence of the periodic forcing, in the Born approximation. Some general examples can be found, for instance, in ref. 38. The idea is to see the time-dependent term in Eq. (5) as the perturbation, i.e., to split the operator of Eq. (5) as

$$L = L_0 + L_I = \frac{\partial}{\partial x}\left\{\sin x - \alpha + D\frac{\partial}{\partial x}\right\} - A\cos(\omega t)\frac{\partial}{\partial x} \tag{13}$$

After discarding an inhomogeneous term, irrelevant in the large-t limit, one obtains the perturbed equilibrium distribution as

$$P_A(x, t) = \int_0^t e^{L_0(t-s)} L_I(s)\, P(x)\, ds \tag{14}$$

or, more explicitly,

$$P_A(x, t) = -A \int_0^t e^{L_0(t-s)} \cos(\omega s) \frac{\partial}{\partial x} P(x)\, ds \tag{15}$$

where $P(x)$ is the equilibrium distribution associated to L_0.

Given that we have a bias in the potential, the motion can escape to infinity in one of two possible directions. We look then for an equilibrium distribution in the sense that we assume that there will be a constant flux of particles coming from infinity which replenishes the system as it is depleted. Under these assumptions the problem has been solved, for instance, in ref. 1, and the equilibrium distribution reads

$$P_{st}(x) = e^{-V(x)/D} \left\{ N - \frac{C}{D} \int_0^x e^{V(s)/D}\, ds \right\} \tag{16}$$

where N and C are determined from normalization and periodicity conditions.

Now we want to, say, compute the response associated to $\sin(x)$ when the periodic forcing is switched on. This means that we must average $\sin(x)$ over the distribution of Eq. (15). This yields

$$\langle \sin(x) \rangle_A = -A \int_0^{2\pi} \sin(x) \int_0^t e^{L_0(t-s)} \cos(\omega s) \frac{\partial}{\partial x} P(x)\, ds\, dx \tag{17}$$

with the understanding, however, that the inner integral is really the propagator of the distribution obtained by applying the operator $L_I(s)$ to the equilibrium distribution.

After some simple algebra, exchanging the integrals, using the operator $L_0^\dagger$ conjugate to L_0 to "move" $\sin(x)$, and finally applying L_I to $P(x)$, we obtain

$$\langle \sin(x) \rangle_A = -\frac{A}{D} \int_0^t \cos(\omega s) \int_0^{2\pi} \sin(x)(t-s) \left\{ -\frac{\partial U}{\partial x} P(x) - C \right\} ds\, dx \tag{18}$$

So far, no assumptions have been made.

We rewrite Eq. (18) as

$$\langle \sin(x) \rangle_A = -\frac{A}{D} \int_0^t \cos(\omega s) \int_0^{2\pi} \sin(x) \left\{ -\frac{\partial U}{\partial x} P(x) - C \right\} We^{-\lambda(t-s)}\, ds\, dx \tag{19}$$

which obviously leads to the introduction of a kind of "size" R of the response,

$$R=\int_0^{2\pi}\left\{-\frac{\partial U}{\partial x}P(x)-C\right\}\sin(x)\,dx \tag{20}$$

and W is some factor which will depend, partly, on the exact dynamics.

It would be possible, in principle, to compute exactly the quantities appearing in Eq. (18), for instance, via the continued-fraction method of ref. 1. On the other hand, we can try to give a rough estimate of these quantities. The simplest approach would be to replace the system with a much simpler bistable system. Obviously the system we are studying is a multistable system, and we then expect that if (for instance, when the noise intensity or the bias is large) the diffusion over more then one well becomes relevant, our approximation will turn out to be not very accurate.

Introduce then the two Kramers rates corresponding to the escape from one well to, respectively, the lower well λ_- and the upper well λ_+. With simple algebra, defining $\lambda=2(\lambda_-+\lambda_+)$, we find that

$$\int_0^{2\pi}\sin(x)\left\{-\frac{\partial U}{\partial x}P(x)-C\right\}We^{-\lambda(t)}\,dx=R\frac{4\lambda_-\lambda_+}{(\lambda_-+\lambda_+)^2}e^{-\lambda t} \tag{21}$$

It is now straightforward to evaluate the quantities apparing in Eq. (21) and to derive a response function for the system.

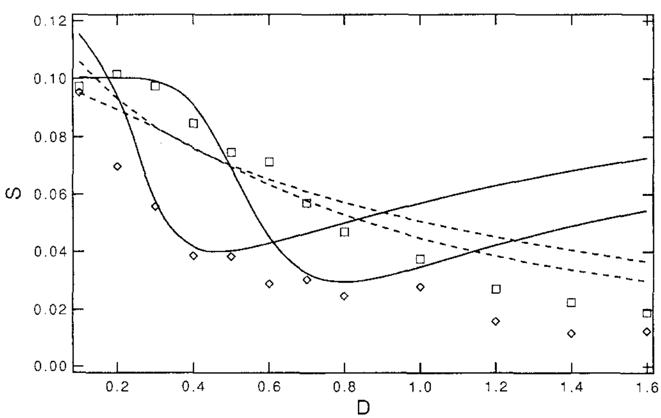

Fig. 4. Signal at the external driving frequency vs. noise. Symbols are numerical simulations for different values of α ($\square=0.1$; $\diamond=0.5$), solid curves are theoretical predictions using both the motion at the bottom of the wells and the Kramers escape between wells, and dashed lines are theoretical predictions using only the motion at the bottom of the wells.

The comparison between theory and the result of numerical simulations relative to the response of $\sin(x)$ is shown in Fig. 4 (solid lines): the agreement is not very satisfactory, and in particular it becomes worse as the noise intensity or the bias is increased. A possible explanation of the disagreement could be that the periodic nature of $\sin(x)$ implies that the important phenomenon is not so much the activation process between the potential wells (there is invariance under $x \to x + 2\pi$) as it is the motion within each well. The theory employing only the oscillations at the bottom of the wells (still taking into account the "size" factor R) is shown as the dashed lines in Fig. 4. Work is under way to derive a better approximation for the correlation function appearing in Eq. (18).

6. CONCLUSIONS

We have investigated the phenomenon of stochastic resonance for a Brownian particle moving in a biased periodic potential. The physical system we used is a phase-locked loop. We briefly discussed a problem natural to the system, i.e., that the response of the variable coupled to the external driving is ill defined. We then turned to two other possible physical quantities, looking for the response of either the velocity or the deterministic force. We showed that when the response of the deterministic forcing ($\sin x$) is considered, the periodic external driving seems unable to give an increase of the signal versus noise intensity. On the other hand, when the velocity is considered we observe a strong increase of the signal at the external driving frequency with increasing noise. Analogies and differences between the present model and the standard bistable system have been discussed, with particular attention to the phase lag between signal and external periodic forcing. Finally, we attempted to understand the behavior of the system in terms of linear response theory, but the agreement between simulations and theory, possibly due to the crude approximations put forward, is far from satisfactory. Theoretical work to refine our approximations is currently under way.

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REFERENCES

1. H. Risken, *The Fokker–Planck Equation, Methods of Solution and Applications* (Springer, 1984).
2. F. Moss and P. V. E. McClintock, eds., *Noise in Nonlinear Dynamical Systems,* Vol. 1. *Theory of Continuous Fokker–Planck Systems* (Cambridge University Press, Cambridge, 1989).
3. F. Moss and P. V. E. McClintock, eds., *Noise in Nonlinear Dynamical Systems,* Vol. 2. *Theory of Noise Induced Processes in Special Applications* (Cambridge University Press, Cambridge, 1989).
4. A. Igarashi, P. V. E. McClintock, and N. G. Stocks, Velocity spectrum for non-Markovian Brownian motion in a periodic potential, *J. Stat. Phys.*, submitted.
5. R. Mannella and P. Grigolini, *Phys. Rev. B* **39**:4722 (1989).
6. R. Benzi, A. Sutera, and A. Vulpiani, *J. Phys. A* **14**:L453 (1981).
7. R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani, *Tellus* **34**:10 (1982).
8. C. Nicolis, *Tellus* **34**:1 (1982).
9. B. Caroli, C. Caroli, B. Roulet, and D. Saint-James, *Physica A* **108**:233 (1981).
10. S. Fauve and F. Heslot, *Phys. Lett.* **97A**:5 (1983).
11. B. McNamara, K. Wiesenfeld, and R. Roy, *Phys. Rev. Lett.* **60**:2626 (1988).
12. B. McNamara and K. Wiesenfeld, *Phys. Rev. A* **39**:4854 (1989).
13. G. Debnath, T. Zhou, and F. Moss, *Phys. Rev. A* **00**:4323 (1989).
14. R. F. Fox, *Phys. Rev. A* **39**:4148 (1989).
15. L. Gammaitoni, F. Marchesoni, E. Menichella-Saetta, and S. Santucci, *Phys. Rev. Lett.* **62**:349 (1989).
16. L. Gammaitoni, E. Menichella-Saetta, S. Santucci, F. Marchesoni, and C. Presilla, *Phys. Rev. A* **40**:2114 (1989).
17. C. Presilla, F. Marchesoni, and L. Gammaitoni, *Phys. Rev. A* **40**:2105 (1989).
18. L. Gammaitoni, M. Martinelli, L. Pardi, and S. Santucci, *Phys. Rev. Lett.* **67**:1799 (1991).
19. L. Gammaitoni, F. Marchesoni, M. Martinelli, L. Pardi, and S. Santucci, *Phys. Lett. A* **158**:449 (1991).
20. H. Gang, G. Nicolis, and C. Nicolis, *Phys. Rev. A* **42**:2030 (1990).
21. P. Jung, *Z. Phys. B* **76**:521 (1989).
22. P. Jung and P. Hänggi, *Europhys. Lett.* **8**:505 (1989).
23. M. Dykman, P. V. E. McClintock, R. Mannella,, and N. G. Stocks, *JETP Lett.* **52**:143 (1990).
24. M. Dykman, R. Mannella, P. V. E. McClintock, and N. G. Stocks, *Phys. Rev. Lett.* **65**:2606 (1990).
25. M. Dykman, R. Mannella, P. V. E. McClintock, and N. G. Stocks, *Phys. Rev. Lett.* **68**:2985 (1992).
26. M. Dykman, G. P. Golubev, D. G. Luchinsky, A. L. Velikovich, and S. V. Tsuprikov, *Sov. Phys. JETP Lett.* **53**:193 (1991).
27. F. Moss, Stochastic resonance: From the ice ages to the monkey's ear, in *Some Problems in Statistical Physics*, G. H. Weiss, ed. (SIAM, Philadelphia, 1992).
28. A. Huberman, J. P. Crutchfield, and N. Packard, *Phys. Rev. Lett.* **37**:750 (1980).
29. N. F. Peterson and A. Davidson, *Appl. Phys. Lett.* **39**:830 (1981).
30. D. D'Humieres, M. R. Beasley, B. A. Huberman, and A. Libchaber, *Phys. Rev. A* **26**:3483 (1982).
31. M. Cirillo and N. F. Pedersen, *Phys. Lett.* **90A**:150 (1982).
32. Z. D. Genchev, Z. G. Ivanov, and B. N. Todorov, *IEEE Trans. Circuits System* **CAS-30**:633 (1983).

33. F. Moss and P. V. E. McClintock, eds., *Noise in Nonlinear Dynamical Systems*, Vol. 3. *Experiments and Simulations* (Cambridge University Press, Cambridge, 1989).
34. S. Faetti, C. Festa, L. Fronzoni, P. Grigolini, and P. Martano, *Phys. Rev. A* **30**:3253 (1984).
35. N. G. Stocks, N. D. Stein, and P. V. E. McClintock, Stochastic resonance in monostable system, unpublished.
36. M. I. Dykman, R. Mannella, P. V. E. McClintock, N. D. Stein, and N. G. Stocks, High frequency stochastic resonance for periodic attractors, unpublished.
37. M. I. Dykman, D. G. Luchinsky, P. V. E. McClintock, N. D. Stein, and N. G. Stocks, *Phys. Rev. A* **46**:R1713 (1992).
38. R. Kubo, *J. Phys. Soc. Japan* **12**:570 (1957).